Errata and comments on ”Generic orthogonal moments: Jacobi–Fourier moments for invariant image description”

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Abstract

Ping *et al.* [Z. Ping, H. Ren, J. Zou, Y. Sheng, and W. Bo, “Generic orthogonal moments: Jacobi–Fourier moments for invariant image description,” *Pattern Recognition*, vol. 40, no. 4, pp. 1245–1254, 2007] made a landmark contribution to the theory of two-dimensional orthogonal moments confined to the unit disk by unifying the radial kernels of existing polynomial-based circular orthogonal moments under the roof of shifted Jacobi polynomials. However, the work contains some errata that result mainly from the confusion between the two slightly different definitions of shifted Jacobi polynomials in the literature. Taking into account the great importance and the high impact of the work in the pattern recognition community, this paper points out the confusing points, corrects the errors, and gives some other relevant comments. The corrections developed in this paper are illustrated by some experimental evidence.

Keywords: Jacobi polynomials, Legendre polynomials, Chebyshev polynomials, Zernike moments, pseudo-Zernike moments, orthogonal Fourier–Mellin moments, Chebyshev–Fourier moments, pseudo-Jacobi–Fourier moments

1 Introduction

Rotation-invariant features of images are usually extracted by using moment methods [1] in which an image f on the unit circle is decomposed into a set of kernels $\{V_{nm} \mid n, m \in \mathbb{Z}\}$ as

$$H_{nm} = \iint\limits_{x^2+y^2\leq 1} f(x,y) V^*_{nm}(x,y)\,\mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\!\!\int_0^1 f(r,\theta)V^*_{nm}(r,\theta) r\,\mathrm{d}r\mathrm{d}\theta,$$

where $x = \cos r, y = \sin r$, $V_{nm}(r,\theta) = R_n(r)A_m(\theta)$ with $A_m(\theta) = \mathrm{e}^{im\theta}$ [2], an asterisk denoted the complex conjugate, and R_n could be of any form. For example, rotational moments (RM) [3] and complex moments (CM) [4] are defined by using r^n for $R_n(r)$. However, the obtained kernels V_{nm} of RM and CM are not orthogonal and, as a result, information redundancy exists in the extracted moments H_{nm}, leading to undesirable effects in image reconstruction and recognition. Orthogonality among kernels means:

$$\langle V_{nm}(x,y), V_{n'm'}(x,y)\rangle = \int_0^1 R_n(r)R^*_{n'}(r) r\,\mathrm{d}r \int_0^{2\pi} A_m(\theta)A^*_{m'}(\theta)\,\mathrm{d}\theta = \delta_{nn'}\delta_{mm'}.$$

From the orthogonality of the circular kernels:

$$\int_0^{2\pi} A_m(\theta)A^*_{m'}(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \mathrm{e}^{im\theta}\mathrm{e}^{-im'\theta}\,\mathrm{d}\theta = 2\pi\delta_{mm'}, \tag{1}$$

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the remaining condition for the radial kernels is:

$$\int_0^1 R_n(r) R_{n'}^*(r) r \,\mathrm{d}r = \frac{1}{2\pi}\,\delta_{nn'}. \tag{2}$$

There exists a number of methods that satisfy the above condition. One direction employs polynomials of the variable r for $R_n(r)$. Popular methods are Zernike moments (ZM) [5], pseudo-Zernike moments (PZM) [3], orthogonal Fourier-Mellin moments (OFMM) [6], *etc* (see [7] for a comprehensive survey). Another direction uses exponential and harmonic functions [8, 9] for R_n. The last direction employs eigenfunctions of the Laplacian on the unit disk [10] for V_{nm}. Interestingly, the radial kernels of methods in the first direction have been shown to be special cases of shifted Jacobi polynomials [11]. This is a landmark contribution to the theory of two-dimensional orthogonal moments confined to the unit disk since, for the first time, the radial kernels of existing polynomial-based circular orthogonal moments are unified under the roof of shifted Jacobi polynomials. This contribution allows the definition of an arbitrary number of unit disk-based orthogonal moment types by choosing different cases of shifted Jacobi polynomials, although the applicability of each case needs further investigation. However, the work contains some errata that result mainly from the confusion between the two slightly different definitions of shifted Jacobi polynomials in the literature. Taking into account the great importance and the high impact of the work in the pattern recognition community, this paper points out the confusing points, corrects the errors, and gives some other relevant comments. The corrections developed in this paper are supported by some experimental evidence.

The remainder of this paper is organized as follows. Section 2 provides the two definitions of shifted Jacobi polynomials, their interconversion, and their relation with Jacobi polynomials. The definition of the radial kernel of generic Jacobi–Fourier moments is redeveloped in Section 3 with explicit formula for some special cases. Section 4 provides the definition of existing polynomial-based radial kernels and develops their relation with shifted Jacobi polynomials. Errata and comments on [11] are then provided in Section 5 by summarizing the relations obtained in the previous sections. Experimental results are given in Section 6, and finally conclusions are drawn in Section 7.

2 Shifted Jacobi polynomials

2.1 Two different definitions

There exist two slightly different definitions of shifted Jacobi polynomials in the literature, one in [12, Chapter 22] and the other in [2], that may make the readers confused since they both satisfy the relation

$$\int_0^1 G_n(p,q,r)\,G_{n'}(p,q,r)\,w(p,q,r)\,\mathrm{d}r = b_n(p,q)\,\delta_{nn'}, \tag{3}$$

where G_n stands for the shifted Jacobi polynomials of order $n \in \mathbb{N}$; w for the weighting function with

$$w(p,q,r) = (1-r)^{p-q}\,r^{q-1} \qquad (p-q > -1, q > 0), \tag{4}$$

b_n for the normalization constants, and $\delta_{nn'} = [n = n']$ for the Kronecker delta function. In order to distinguish the two definitions, we use superscripts

- A for the definition coming from [12] such as G_n^A and b_n^A (defined at pages 774–775 of [12]) and
- B for the definition coming from [2] such as G_n^B and b_n^B (defined at page 45 of [2]).

The original definitions of G_n^A, b_n^A, G_n^B, and b_n^B in the above two sources are listed in Table 1 for convenience. It can be seen that, due to the use of factorial functions in the case of B, the values of p and q are more limited than the conditions in equation (4) so that G_n^B and b_n^B are both defined. In fact, p and q are additionally required to take integer values. However, this integral restriction on p and q can be easily overcome by using Gamma functions [12, Chapter 6] to have the following generic forms of G_n^B and b_n^B:

$$G_n^B(p,q,r) = \frac{n!\,\Gamma(q)}{\Gamma(p+n)} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(p+n+k)}{(n-k)!\,k!\,\Gamma(q+k)}\, r^k, \tag{5}$$

Table 1: Two different definitions of shifted Jacobi polynomials and their corresponding normalization constants. G_n^A and b_n^A are from [12] whereas G_n^B and b_n^B are from [2].

Shifted Jacobi polynomials	Normalization constants
$G_n^A(p,q,r) = \frac{\Gamma(q+n)}{\Gamma(p+2n)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(p+2n-k)}{\Gamma(q+n-k)} r^{n-k}$	$b_n^A(p,q) = \frac{n!\,\Gamma(n+q)\,\Gamma(n+p)\,\Gamma(n+p-q+1)}{(2n+p)\,\Gamma^2(2n+p)}$
$G_n^B(p,q,r) = \frac{n!\,(q-1)!}{(p+n-1)!} \sum_{k=0}^{n} (-1)^k \frac{(p+n+k-1)!}{(n-k)!\,k!\,(q+k-1)!} r^k$	$b_n^B(p,q) = \frac{n!\,\left[(q-1)!\right]^2 (p-q+n)!}{(q+n-1)!\,(p+n-1)!\,(p+2n)}$

$$b_n^B(p,q) = \frac{n!\,\Gamma^2(q)\,\Gamma(p-q+n+1)}{\Gamma(q+n)\,\Gamma(p+n)\,(p+2n)}. \tag{6}$$

Since $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, these generic forms using Gamma functions reduce to the non-generic forms using factorial functions in Table 1 when $p, q \in \mathbb{N}$. In the remaining of this paper, unless explicitly stated otherwise, the generic forms in equations (5) and (6) are used by default whenever G_n^B or b_n^B are referred to.

2.2 Interconversion between the two definitions

In spite of their different looks, G_n^A and G_n^B are in fact related as demonstrated below:

$$\begin{aligned}
G_n^A(p,q,r) &= \frac{\Gamma(q+n)}{\Gamma(p+2n)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(p+2n-k)}{\Gamma(q+n-k)} r^{n-k} \\
&= \frac{\Gamma(q+n)}{\Gamma(p+2n)} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{\Gamma(p+n+k)}{\Gamma(q+k)} r^{k} \\
&= \frac{\Gamma(q+n)}{\Gamma(p+2n)} \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{(n-k)!\,k!} \frac{\Gamma(p+n+k)}{\Gamma(q+k)} r^{k} \\
&= \frac{\Gamma(q+n)\,\Gamma(p+n)}{\Gamma(p+2n)\,\Gamma(q)} \frac{n!\,\Gamma(q)}{\Gamma(p+n)} \sum_{k=0}^{n} (-1)^{n-k} \frac{\Gamma(p+n+k)}{(n-k)!\,k!\,\Gamma(q+k)} r^{k} \\
&= (-1)^n \frac{\Gamma(q+n)\,\Gamma(p+n)}{\Gamma(p+2n)\,\Gamma(q)} G_n^B(p,q,r).
\end{aligned} \tag{7}$$

Similarly, for b_n^A and b_n^B:

$$\begin{aligned}
\frac{b_n^A(p,q)}{b_n^B(p,q)} &= \frac{n!\,\Gamma(n+q)\,\Gamma(n+p)\,\Gamma(n+p-q+1)}{(2n+p)\,\Gamma^2(2n+p)} \left[\frac{n!\,\Gamma^2(q)\,\Gamma(p-q+n+1)}{\Gamma(q+n)\,\Gamma(p+n)\,(p+2n)}\right]^{-1} \\
&= \left[\frac{\Gamma(n+q)\,\Gamma(n+p)}{\Gamma(2n+p)\,\Gamma(q)}\right]^2.
\end{aligned} \tag{8}$$

From equations (7) and (8) we have:

$$\left[\frac{1}{b_n^A(p,q)}\right]^{\frac{1}{2}} G_n^A(p,q,r) = (-1)^n \left[\frac{1}{b_n^B(p,q)}\right]^{\frac{1}{2}} G_n^B(p,q,r). \tag{9}$$

2.3 Relation with Jacobi polynomials

Shifted Jacobi polynomials are also related to Jacobi polynomials [12, Chapter 22]. Equations 22.5.1 and 22.5.2 at page 777 of [12] state that

$$P_n^{(p,q)}(r) = \frac{\Gamma(2n+p+q+1)}{n!\,\Gamma(n+p+q+1)} G_n^A\left(p+q+1, q+1, \frac{r+1}{2}\right),$$

$$G_n^A(p,q,r) = \frac{n!\,\Gamma(n+p)}{\Gamma(2n+p)}\,P_n^{(p-q,q-1)}(2r-1),$$

where $P_n^{(p,q)}$ stands for Jacobi polynomials. Combining the above two equations with equation (7) leads to

$$\begin{aligned}
P_n^{(p,q)}(r) &= \frac{\Gamma(2n+p+q+1)}{n!\,\Gamma(n+p+q+1)}\,G_n^A\left(p+q+1, q+1, \frac{r+1}{2}\right) \\
&= \frac{\Gamma(2n+p+q+1)}{n!\,\Gamma(n+p+q+1)}\,(-1)^n\,\frac{\Gamma(q+1+n)\,\Gamma(p+q+1+n)}{\Gamma(p+q+1+2n)\,\Gamma(q+1)}\,G_n^B\left(p+q+1, q+1, \frac{r+1}{2}\right) \\
&= (-1)^n\,\frac{\Gamma(n+q+1)}{n!\,\Gamma(q+1)}\,G_n^B\left(p+q+1, q+1, \frac{r+1}{2}\right) \qquad (10)\\
G_n^B(p,q,r) &= (-1)^n\,\frac{\Gamma(p+2n)\,\Gamma(q)}{\Gamma(q+n)\,\Gamma(p+n)}\,G_n^A(p,q,r) \\
&= (-1)^n\,\frac{\Gamma(p+2n)\,\Gamma(q)}{\Gamma(q+n)\,\Gamma(p+n)}\,\frac{n!\,\Gamma(n+p)}{\Gamma(2n+p)}\,P_n^{(p-q,q-1)}(2r-1) \\
&= (-1)^n\,\frac{n!\,\Gamma(q)}{\Gamma(n+q)}\,P_n^{(p-q,q-1)}(2r-1).
\end{aligned}$$

3 The radial kernels J_n of generic Jacobi–Fourier moments

3.1 Definition of J_n

Equation (9) means that G_n^A and G_n^B are equivalent and differ by a constant multiplicative factor, $(-1)^n\left[\frac{b_n^A}{b_n^B}\right]^{1/2}$, and that J_n in equation (8) of [11] can be defined by using one of the following two equations:

$$\begin{aligned}
J_n^A(p,q,r) &= \left[\frac{w(p,q,r)}{r\,b_n^A(p,q)}\right]^{\frac{1}{2}} G_n^A(p,q,r) \qquad \text{or} \\
J_n^B(p,q,r) &= \left[\frac{w(p,q,r)}{r\,b_n^B(p,q)}\right]^{\frac{1}{2}} G_n^B(p,q,r).
\end{aligned}$$

It is then straightforward from equation (9) that

$$J_n^A(p,q,r) = (-1)^n\,J_n^B(p,q,r). \qquad (11)$$

The interleaving difference in sign between J_n^A and J_n^B does not affect the nature of the representation. It, however, only affects the sign of the representation coefficients. In addition, from equation (3):

$$\begin{aligned}
&\int_0^1 J_n^A(p,q,r)\,\left[J_{n'}^A(p,q,r)\right]^*\,r\,\mathrm{d}r = \delta_{nn'} \qquad (12)\\
&\int_0^1 J_n^B(p,q,r)\,\left[J_{n'}^B(p,q,r)\right]^*\,r\,\mathrm{d}r = \delta_{nn'}.
\end{aligned}$$

Thus, J_n^A and J_n^B can be used to define radial kernels R_n in equation (2).

3.2 Some special cases of J_n

For $p = 2$, $q = 2$

$$\begin{aligned}
w(2,2,r) &= r \\
b_n^B(2,2) &= \frac{n!\,n!}{(n+1)!\,(n+1)!\,(2n+2)} \\
&= \frac{1}{2\,(n+1)^3}
\end{aligned}$$

$$G_n^B(2,2,r) = \frac{n!}{(n+1)!} \sum_{k=0}^{n} (-1)^k \frac{(n+k+1)!}{(n-k)!\,k!\,(k+1)!}\, r^k$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \frac{(n+k+1)!}{(n-k)!\,k!\,(k+1)!}\, r^k \tag{13}$$

$$J_n^B(2,2,r) = \left[\frac{r\,2\,(n+1)^3}{r}\right]^{\frac{1}{2}} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \frac{(n+k+1)!}{(n-k)!\,k!\,(k+1)!}\, r^k$$

$$= [2\,(n+1)]^{\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \frac{(n+k+1)!}{(n-k)!\,k!\,(k+1)!}\, r^k \tag{14}$$

$$J_n^A(2,2,r) = (-1)^n\,[2\,(n+1)]^{\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \frac{(n+k+1)!}{(n-k)!\,k!\,(k+1)!}\, r^k \tag{15}$$

For $p = 3,\ q = 2$

$$w(3,2,r) = (1-r)\,r$$

$$b_n^B(3,2) = \frac{n!\,(n+1)!}{(n+1)!\,(n+2)!\,(2n+3)}$$

$$= \frac{1}{(n+1)(n+2)(2n+3)}$$

$$G_n^B(3,2,,r) = \frac{n!}{(n+2)!} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\,k!\,(k+1)!}\, r^k$$

$$= \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\,k!\,(k+1)!}\, r^k$$

$$J_n^B(3,2,r) = \left[\frac{(1-r)\,r\,(n+1)(n+2)(2n+3)}{r}\right]^{\frac{1}{2}} \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\,k!\,(k+1)!}\, r^k$$

$$= \left[\frac{(2n+3)}{(n+1)(n+2)}\,(1-r)\right]^{\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\,k!\,(k+1)!}\, r^k \tag{16}$$

$$J_n^A(3,2,r) = (-1)^n \left[\frac{(2n+3)}{(n+1)(n+2)}\,(1-r)\right]^{\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\,k!\,(k+1)!}\, r^k \tag{17}$$

For $p = 3,\ q = 3$

$$w(3,3,r) = r^2$$

$$b_n^B(3,3) = \frac{n!\,\left[2!\right]^2 n!}{(n+2)!\,(n+2)!\,(2n+3)}$$

$$= \frac{4}{(n+1)^2\,(n+2)^2\,(2n+3)}$$

$$G_n^B(3,3,r) = \frac{n!\,2!}{(n+2)!} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\,k!\,(k+2)!}\, r^k$$

$$= \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\,k!\,(k+2)!}\, r^k$$

$$J_n^B(3,3,r) = \left[\frac{r^2\,(n+1)^2\,(n+2)^2\,(2n+3)}{r\,4}\right]^{\frac{1}{2}} \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\,k!\,(k+2)!}\, r^k$$

$$= [(2n+3)\,r]^{\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\,k!\,(k+2)!}\, r^k \tag{18}$$

$$J_n^A(3,3,r) = (-1)^n \left[(2n+3)\, r\right]^{\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \frac{(n+k+2)!}{(n-k)!\, k!\, (k+2)!}\, r^k \tag{19}$$

For $p=4$**,** $q=3$

$$\begin{aligned}
w(4,3,r) &= (1-r)\, r^2 \\
b_n^B(4,3) &= \frac{n!\, \left[2!\right]^2 (n+1)!}{(n+2)!\, (n+3)!\, (2n+4)} \\
&= \frac{2}{(n+1)(n+2)^3\, (n+3)} \\
G_n^B(4,3,r) &= \frac{n!\, 2!}{(n+3)!} \sum_{k=0}^{n} (-1)^k \frac{(n+k+3)!}{(n-k)!\, k!\, (k+2)!}\, r^k \\
&= \frac{2}{(n+1)(n+2)(n+3)} \sum_{k=0}^{n} (-1)^k \frac{(n+k+3)!}{(n-k)!\, k!\, (k+2)!}\, r^k \\
J_n^B(4,3,r) &= \left[\frac{(1-r)\, r^2\, (n+1)(n+2)^3\, (n+3)}{r\, 2}\right]^{\frac{1}{2}} \frac{2}{(n+1)(n+2)(n+3)} \sum_{k=0}^{n} (-1)^k \frac{(n+k+3)!}{(n-k)!\, k!\, (k+2)!}\, r^k \\
&= \left[\frac{2\,(n+2)}{(n+1)(n+3)}\, \left(r-r^2\right)\right]^{\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \frac{(n+k+3)!}{(n-k)!\, k!\, (k+2)!}\, r^k \qquad (20) \\
J_n^A(4,3,r) &= (-1)^n \left[\frac{2\,(n+2)}{(n+1)(n+3)}\, \left(r-r^2\right)\right]^{\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \frac{(n+k+3)!}{(n-k)!\, k!\, (k+2)!}\, r^k \qquad (21)
\end{aligned}$$

4 Polynomial-based radial kernels

4.1 Zernike moments (ZM) and pseudo-Zernike moments (PZM)

For a fixed value of the angular order m, the radial kernels R_n^m of ZM [13] and P_n^m of PZM [2] are defined to be the polynomials of order n that arise out from Gram–Schmidt orthogonalization of the polynomial sequences $\{r^{|m|}, r^{|m|+2}, r^{|m|+4}, r^{|m|+6}, \ldots\}$ and $\{r^{|m|}, r^{|m|+1}, r^{|m|+2}, r^{|m|+3}, \ldots\}$, respectively, with the weighting function r over the range $0 \le r \le 1$. It was shown that R_n^m has the following explicit expression:

$$R_n^m(r) = \sum_{k=0}^{\frac{n-|m|}{2}} (-1)^k \frac{(n-k)!}{k!\, \left(\frac{n+|m|}{2} - k\right)!\, \left(\frac{n-|m|}{2} - k\right)!}\, r^{n-2k},$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ satisfying $n - |m| = \text{even}$ and $|m| \le n$. The explicit expression of P_n^m is

$$P_n^m(r) = \sum_{k=0}^{n-|m|} (-1)^k \frac{(2n+1-k)!}{k!\, (n+|m|+1-k)!\, (n-|m|-k)!}\, r^{n-k},$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ satisfying $|m| \le n$. It is then straightforward that

$$\int_0^1 R_n^m(r) R_{n'm}^*(r)\, r\, \mathrm{d}r = \frac{1}{2n+2}\, \delta_{nn'},$$
$$\int_0^1 P_n^m(r) P_{n'm}^*(r)\, r\, \mathrm{d}r = \frac{1}{2n+2}\, \delta_{nn'}.$$

Relation with G_n^B: The relations between R_n^m and G_n^B, P_n^m and G_n^B are given in equations (3.15) and (4.3) respectively of [2] as

$$R_n^m(r) = (-1)^{\frac{1}{2}(n-|m|)} \binom{\frac{1}{2}(n+|m|)}{\frac{1}{2}(n-|m|)} r^{|m|} G_{\frac{1}{2}(n-|m|)}^B(|m|+1, |m|+1, r^2), \tag{22}$$

$$P_n^m(r) = (-1)^{n-|m|} \binom{n+|m|+1}{n-|m|} r^{|m|} G_{n-|m|}^B(2\,|m|+2, 2\,|m|+2, r). \tag{23}$$

4.2 Orthogonal Fourier–Mellin moments (OFMM)

Similar to ZM and PZM, the radial kernels Q_n of OFMM [6] are obtained by changing the orthogonalizing polynomial sequence to be $\{1, r, r^2, r^3, \ldots\}$ while keeping the weighting function r. It is also not difficult to arrive at the following definition of Q_n with $n \in \mathbb{N}$:

$$Q_n(r) = \sum_{k=0}^{n} (-1)^{n+k} \frac{(n+k+1)!}{(n-k)!\,k!\,(k+1)!} r^k. \tag{24}$$

Note that Q_n does not depend on the angular order m and satisfies the following identity:

$$\int_0^1 Q_n(r) Q_{n'}^*(r)\, r\, \mathrm{d}r = \frac{1}{2n+2}\, \delta_{nn'}. \tag{25}$$

Relation with G_n^B: From equations (13) and (24) of this paper, we have:

$$Q_n(r) = (-1)^n\,(n+1)\, G_n^B(2, 2, r). \tag{26}$$

4.3 Chebyshev–Fourier moments (CHFM)

The radial kernels R_n of CHFM [14] are defined based on shifted Chebyshev polynomials of the second kind $U_n^\star$ [12, Chapter 22] of the same order. By definition, $U_n^\star$ are themselves orthogonal with the weighting function as $w^\star(r) = (r - r^2)^{1/2}$ over the range $0 \le r \le 1$:

$$\int_0^1 U_n^\star(r) \left[U_{n'}^\star(r)\right]^* w^\star(r)\, \mathrm{d}r = \frac{\pi}{8}\, \delta_{nn'}.$$

By using the identity $U_n^\star(r) = U_n(2r-1)$ from equation (22.5.15) at page 778 of [12] and the explicit expression of Chebyshev polynomials of the second kind U_n from equation (22.3.7) at page 775 of [12]

$$U_n(r) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(n-k)!}{k!\,(n-2k)!} (2r)^{n-2k},$$

we arrive at the following explicit expression of $U_n^\star(r)$:

$$U_n^\star(r) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(n-k)!}{k!\,(n-2k)!} (4r-2)^{n-2k}.$$

If R_n is defined as

$$\begin{aligned} R_n(r) &= \left[\frac{8\, w^\star(r)}{\pi\, r}\right]^{\frac{1}{2}} U_n^\star(r) \\ &= \left[\frac{64\,(1-r)}{\pi^2\, r}\right]^{\frac{1}{4}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(n-k)!}{k!\,(n-2k)!} (4r-2)^{n-2k}, \end{aligned}$$

it is then straightforward that

$$\int_0^1 R_n(r) R_{n'}^*(r)\, r\, \mathrm{d}r = \delta_{nn'}.$$

Relation with G_n^B: Using equation (10) of this paper and equation (22.5.32) at page 778 of [12] to relate U_n to G_n^B as

$$\begin{aligned} U_n(r) &= \frac{(n+1)!\,\sqrt{\pi}}{2\,\Gamma\big(n+\frac{3}{2}\big)}\, P_n^{(\frac{1}{2},\frac{1}{2})}(r) \\ &= \frac{(n+1)!\,\sqrt{\pi}}{2\,\Gamma\big(n+\frac{3}{2}\big)}\,(-1)^n\,\frac{\Gamma\big(n+\frac{3}{2}\big)}{n!\,\Gamma\big(\frac{3}{2}\big)}\,G_n^B\left(2,\frac{3}{2},\frac{r+1}{2}\right) \\ &= (-1)^n\,\frac{(n+1)\,\sqrt{\pi}}{2\,\Gamma\big(\frac{3}{2}\big)}\,G_n^B\left(2,\frac{3}{2},\frac{r+1}{2}\right) \\ &= (-1)^n\,(n+1)\,G_n^B\left(2,\frac{3}{2},\frac{r+1}{2}\right), \end{aligned} \tag{27}$$

since $\Gamma\big(\frac{3}{2}\big)=\frac{\sqrt{\pi}}{2}$. Then we have:

$$\begin{aligned} R_n(r) &= \left[\frac{8\,w^\star(r)}{\pi\,r}\right]^{\frac{1}{2}}\,U_n^\star(r) \\ &= \left[\frac{64\,(1-r)}{\pi^2\,r}\right]^{\frac{1}{2}}\,U_n(2r-1) \\ &= (-1)^n\left[\frac{64\,(1-r)}{\pi^2\,r}\right]^{\frac{1}{4}}\,(n+1)\,G_n^B\left(2,\frac{3}{2},r\right). \end{aligned} \tag{28}$$

4.4 Pseudo Jacobi–Fourier moments (PJFM)

Similar to CHFM, the radial kernels R_n of PJFM [15] are defined based on shifted Jacobi polynomials G_n^A of the same order using the following explicit expression:

$$R_n(r) = \left[\frac{2n+4}{(n+3)(n+1)}\,\big(r-r^2\big)\right]^{\frac{1}{2}}\sum_{k=0}^{n}(-1)^{n+k}\,\frac{(n+k+3)!}{(n-k)!\,k!\,(k+2)!}\,r^k, \tag{29}$$

which leads to the identity:

$$\int_0^1 R_n(r)R_{n'}^*(r)\,r\,\mathrm{d}r = \delta_{nn'}.$$

Relation with G_n^B: From equations (20) and (29) of this paper, we have:

$$\begin{aligned} R_n(r) &= (-1)^n\,J_n^B(4,3,r) \\ &= (-1)^n\left[\frac{w(4,3,r)}{r\,b_n^B(4,3)}\right]^{\frac{1}{2}}\,G_n^B(4,3,r) \\ &= (-1)^n\left[\frac{(n+1)(n+2)^3(n+3)}{2}\,\big(r-r^2\big)\right]^{\frac{1}{2}}\,G_n^B(4,3,r). \end{aligned} \tag{30}$$

5 Errata and comments

5.1 Definitions of G_n^B and b_n^B

Although [11] uses G_n^B and b_n^B as the shifted Jacobi polynomials and the corresponding normalization constants in its equations (4) and (6), it incorrectly cites [12] as the source of information, instead of [2]. This is incorrect because of the difference between the two definitions of shifted Jacobi polynomials and normalization constants in [12] and [2] that have been demonstrated in Section 2. Nevertheless, the explicit expressions and the use of G_n^B and b_n^B in [11] indicate that [11] adopts the definition of shifted

Jacobi polynomials and the corresponding normalization constants from [2], not from [12]. Equation (11) of this paper implies

$$J_n(p,q,r) = J_n^B(p,q,r) = (-1)^n\, J_n^A(p,q,r), \tag{31}$$

where $J_n(p,q,r)$ is defined in equation (8) of [11]. The corrections and comments that follow in the remaining of this section result mainly from the confusing use of G_n^A, b_n^A and G_n^B, b_n^B in [11].

5.2 Some special cases of J_n

Comparing the explicit expressions of $J_n^A(2,2,r)$, $J_n^A(3,2,r)$, $J_n^A(3,3,r)$, and $J_n^A(4,3,r)$ given at equations (15), (17), (19), and (21) of this paper and $J_n(2,2,r)$, $J_n(3,2,r)$, $J_n(3,3,r)$, and $J_n(4,3,r)$ at equations (11)–(14) of [11] leads to the following observations:

- The multiplicative term $[2\,(n+1)]^{\frac{1}{2}}$ is missing in the explicit expression of $J_n(2,2,r)$.
- Other than the missing multiplicative term, it can be easily seen that $J_n(p,q,r) = J_n^A(p,q,r)$. This identity contradicts with the identity in equation (31) of this paper.

Thus, the explicit expressions of $J_n(2,2,r)$, $J_n(3,2,r)$, $J_n(3,3,r)$, and $J_n(4,3,r)$ at equations (11)–(14) of [11] are incorrect. The correct expressions of $J_n(2,2,r)$, $J_n(3,2,r)$, $J_n(3,3,r)$, and $J_n(4,3,r)$ are $J_n^B(2,2,r)$, $J_n^B(3,2,r)$, $J_n^B(3,3,r)$, and $J_n^B(4,3,r)$ respectively given at equations (14), (16), (18), and (20) of this paper.

5.3 Relation with Legendre polynomials and Chebyshev polynomials

Equations (1'), (3'), (5') in Appendix A.1 and equations (7'), (9') in Appendix A.2 of [11] apply for G_n^A, not for G_n^B, since they are taken from [12]. Because [11] provides only the definition for G_n^B in its equation (4) and does not mention G_n^A, the readers may assume that G_n^B is used in those appendix's equations. This assumption then makes these equations incorrect. Thus, an explicit expression of G_n^A and its difference with G_n^B should be provided at the beginning of the appendix or all the appendix equations should be rewritten to use G_n^B, instead of G_n^A, using the relations already developed in this paper. For example, equation (10) of this paper can be used to correct equation (1') in Appendix A.1 of [11] as

$$\begin{aligned} P_n^{(p,q)}(r) &= \frac{\Gamma(2n+p+q+1)}{n!\,\Gamma(n+p+q+1)}\, G_n^A\left(p+q+1, q+1, \frac{r+1}{2}\right) \\ &= (-1)^n\, \frac{\Gamma(n+q+1)}{n!\,\Gamma(q+1)}\, G_n^B\left(p+q+1, q+1, \frac{r+1}{2}\right). \end{aligned}$$

This leads to the following correction for equation (3') in Appendix A.1 of [11]:

$$\begin{aligned} P_n(r) &= P_n^{(0,0)}(r) \\ &= \frac{\Gamma(2n+1)}{n!\,\Gamma(n+1)}\, G_n^A\left(1, 1, \frac{r+1}{2}\right) \\ &= (-1)^n\, \frac{\Gamma(n+1)}{n!}\, G_n^B\left(1, 1, \frac{r+1}{2}\right). \end{aligned}$$

And equation (27) of this paper can be used to correct equation (7') in Appendix A.2 of [11] as

$$\begin{aligned} U_n(r) &= 4^n\, G_n^A\left(2, \frac{3}{2}, \frac{r+1}{2}\right) \\ &= (-1)^n\,(n+1)\, G_n^B\left(2, \frac{3}{2}, \frac{r+1}{2}\right). \end{aligned}$$

Table 2: Relations between the radial kernels of existing polynomial-based circular orthogonal moments and shifted Jacobi polynomials G_n^B. Each of the existing polynomial-based radial kernels is a special case of G_n^B obtained by properly setting the values of the two parameters p and q.

Method	Relationship
ZM	$R_n^m(r) = (-1)^{\frac{1}{2}(n-\lvert m\rvert)} \binom{\frac{1}{2}(n+\lvert m\rvert)}{\frac{1}{2}(n-\lvert m\rvert)} r^{\lvert m\rvert} G^B_{\frac{1}{2}(n-\lvert m\rvert)}(\lvert m\rvert+1, \lvert m\rvert+1, r^2)$
PZM	$P_n^m(r) = (-1)^{n-\lvert m\rvert} \binom{n+\lvert m\rvert+1}{n-\lvert m\rvert} r^{\lvert m\rvert} G^B_{n-\lvert m\rvert}(2\lvert m\rvert+2, 2\lvert m\rvert+2, r)$
OFMM	$Q_n(r) = (-1)^n (n+1) G_n^B(2,2,r)$
CHFM	$R_n(r) = (-1)^n \left[\frac{64\,(1-r)}{\pi^2\, r}\right]^{\frac{1}{4}} (n+1) G_n^B(2, \frac{3}{2}, r)$
PJFM	$R_n(r) = (-1)^n \left[\frac{(n+1)(n+2)^3(n+3)}{2}\left(r-r^2\right)\right]^{\frac{1}{2}} G_n^B(4,3,r)$

5.4 Generic orthogonal moments

According to [2], G_n^B is obtained by orthogonalizing the polynomial sequence $\{1, r, r^2, r^3, \ldots\}$ with the weighting function w over the range $0 \le r \le 1$, similar to the way the radial kernels of ZM, PZM, and OFMM are derived. Moreover, since Chebyshev polynomials are a special case of the Jacobi polynomials as shown in equation (27) of this paper, the radial kernels of CHFM could be defined directly from G_n^B, like in the case of PJFM. These observations lead to the conclusion that all of the aforementioned radial kernels of polynomial-based circular orthogonal moments are special cases of G_n^B [16]. Collecting the identities from equations (22), (23), (26), (28), and (30) of this paper gives:

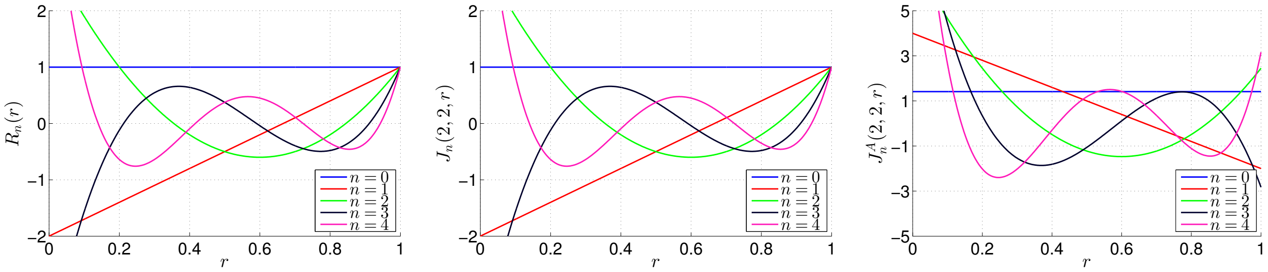
- ZM: $p = \lvert m\rvert + 1$, $q = \lvert m\rvert + 1$
- PZM: $p = 2\lvert m\rvert + 2$, $q = 2\lvert m\rvert + 2$
- OFMM: $p = 2$, $q = 2$
- CHFM: $p = 2$, $q = \frac{3}{2}$
- PJFM: $p = 4$, $q = 3$.

Explicit relations between the radial kernels of existing polynomial-based circular orthogonal moments and shifted Jacobi polynomials G_n^B are given in Table 2. From these relations, it is not difficult to see that the claim $J_n^B(2,2,r) = Q_n(r)$ in Appendix A.3 and equations (10') and (11') in Appendix A.4 of [11] are incorrect.

- The relation mentioned in Appendix A.3 of [11] can be corrected by combining equations (14) and (24) of this paper as

$$J_n^B(2,2,r) = (-1)^n \left[2\,(n+1)\right]^{\frac{1}{2}} Q_n(r).$$

- Equations (10') and (11') in Appendix A.4 of [11] can be corrected by using equations (22) and (23) of this paper respectively.



(a) OFMM (b) JFM (c) JFM$^{\mathrm{A}}$

Figure 1: The radial kernels of JFM, JFM$^{\mathrm{A}}$ for the case $p = 2, q = 2$ and the corresponding method OFMM at $n = 0, 1, 2, 3, 4$. (a) OFMM according to [6]; (b) JFM according to [11]; (c) JFM$^{\mathrm{A}}$ redeveloped in this paper.

5.5 Gamma function-based definitions of G_n^B and b_n^B

[11] defines Jacobi–Fourier moments (JFM) as the generic version of existing polynomial-based circular orthogonal moments. This is possible by allowing the two parameters p and q in the definition of $J_n(p, q, r)$ in equation (8) of [11] to vary. If G_n^B and b_n^B are defined based on factorial functions as in [11] then p and q can then only take integral values. In this case, CHFM is not a special case of JFM as claimed in [14] since p and q should be 2 and $\frac{3}{2}$ respectively. This problem could be overcome by defining G_n^B and b_n^B based on Gamma functions as given in equations (5) and (6) of this paper.

6 Experimental results

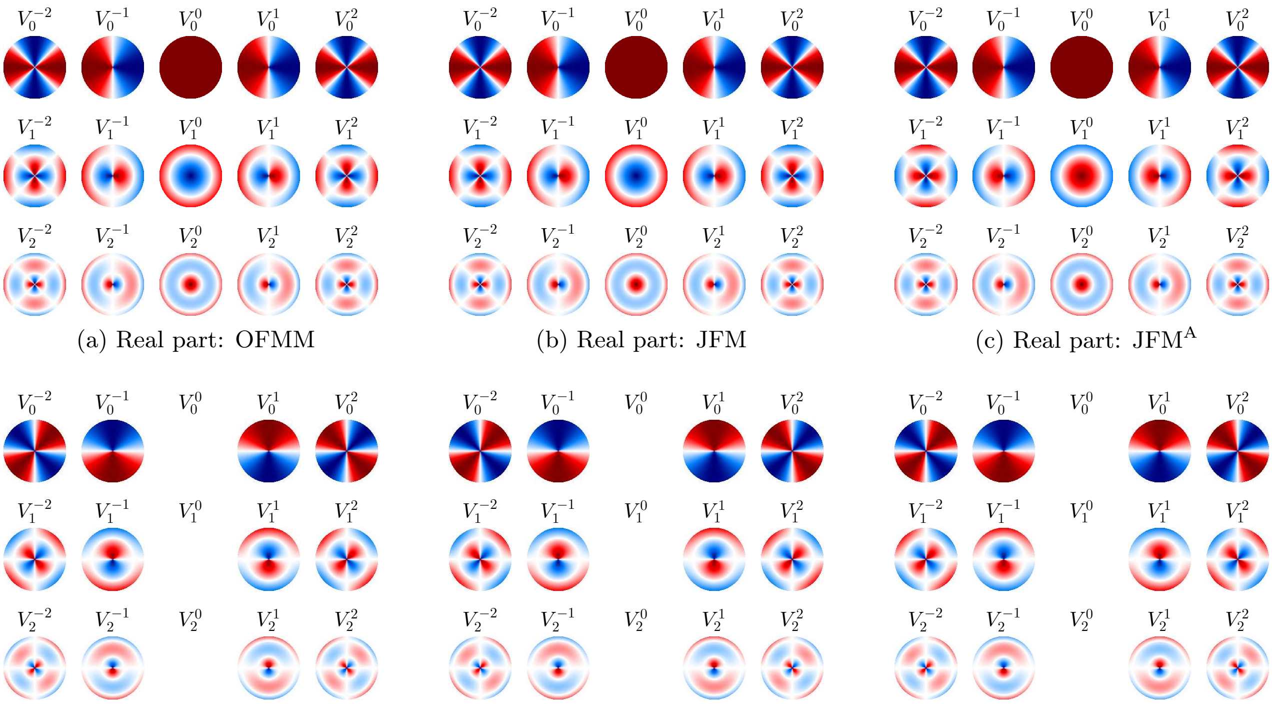
Some experiments have been carried out to elucidate the corrections developed in the previous section. For simplicity, experiments are limited to the case $p = 2, q = 2$ of JFM and the corresponding method OFMM proposed in [6]. However, the generic nature of the formulas of JFM guarantees that similar results can also be observed in other cases of JFM with different values in p and q. In the remaining of this section, JFM denotes the generic moments described in [11] whereas JFM$^{\mathrm{A}}$ denotes the generic moments redeveloped in this paper using the definition of shifted Jacobi polynomials in [12].

The first experiment concerns the form of radial kernels. Figure 1 shows the radial kernels of JFM, JFM$^{\mathrm{A}}$ for the case $p = 2, q = 2$ and the corresponding method OFMM at $n = 0, 1, 2, 3, 4$. It can be observed that $R_n(r)$ has the same form as that of $J_n(2, 2, r)$ at all n. This is because the definition of $R_n(r)$ in equation (24) is identical to the definition of $J_n(2, 2, r)$ in equation (11) of [11]. For the radial kernel $J_n^A(2, 2, r)$, it can also be seen that $J_n^A(2, 2, r)$ differs from $J_n(2, 2, r)$ in the following two ways:

- The interleaving difference in sign, which agrees with equation (31) of this paper.
- The difference in magnitude, which agrees with the observation on the missing multiplicative term $[2\,(n + 1)]^{\frac{1}{2}}$ in Section 5.2 of this paper.

The similar/difference in radial kernels of OFMM, JFM, and JFM$^{\mathrm{A}}$ will lead to the similar/difference in their kernel functions in the experiments that follow.

The effect of interleaving difference in the sign of radial kernels above can also be seen in the 2D view of kernel functions. Figure 2 provides the 2D views of the real and imaginary parts of the kernel functions of JFM, JFM$^{\mathrm{A}}$ for the case $p = 2, q = 2$ and the corresponding method OFMM at $n = 0, 1, 2$ and $m = -2, -1, 0, 1, 2$. Again, it can be observed that the color pattern of the kernel functions of OFMM is identical to that of the kernel functions of JFM. For the kernel functions of JFM$^{\mathrm{A}}$, their color pattern is interleaving reversed with respect to the radial order n when compared to the color pattern of the kernel functions of JFM. For example, in the two Figures 2(b) and 2(c), the color pattern of corresponding images in their first and third rows is the same whereas that of corresponding images in their second row are reversed. These two observations conform with the observations on the form of radial kernels in the previous experiment. The interleaving reversed color pattern means that the



(a) Real part: OFMM (b) Real part: JFM (c) Real part: JFM$^{\text{A}}$

(d) Imaginary part: OFMM (e) Imaginary part: JFM (f) Imaginary part: JFM$^{\text{A}}$

Figure 2: The 2D views of the real and imaginary parts of the kernel functions of JFM, JFM$^{\text{A}}$ for the case $p = 2, q = 2$ and the corresponding method OFMM at $n = 0, 1, 2$ and $m = -2, -1, 0, 1, 2$ using the "blue-white-red" colormap. (a),(d) OFMM according to [6]; (b),(e) JFM according to [11]; (c),(f) JFM$^{\text{A}}$ redeveloped in this paper.

corresponding kernel functions of JFM and JFM$^{\text{A}}$ have interleaving difference in sign, which results in the interleaving difference in the sign of the computed value of moments when these kernel functions are applied on the same image. Note that since the color pattern is plotted with normalized colormap, the effect of missing multiplicative term cannot be observed from Figure 2.

The effect of missing multiplicative term can be seen in the image reconstruction experiment. In order to reconstruct an image from its moments at a certain value of K, all moments H_{nm} of orders (n, m) satisfying $0 \leq |m|, n \leq K$ are computed and then used to reconstruct the image [16]. Figure 3 shows the reconstructed images for the character image "E" of size 64×64 for $K = 0, 1, \ldots, 10$ by JFM, JFM$^{\text{A}}$ for the case $p = 2, q = 2$ and the corresponding method OFMM. It can be observed that the images reconstructed by OFMM and JFM$^{\text{A}}$ are the same, even though OFMM and JFM$^{\text{A}}$ have different forms of radial kernels. This is because OFMM and JFM$^{\text{A}}$ have different normalization factors for their radial kernels, which is $\frac{1}{2n+2}$ in OFMM (equation (25)) and 1 in JFM$^{\text{A}}$ (equation (12)). Since JFM also has normalization factor 1 for its radial kernels due to the definition of J_n in equation (8) of [11], the images reconstructed by JFM are incorrect as they cannot asymptotically reflect the original character image "E" when K increases.

The above three experiments have shown the effects of incorrect definition of the radial kernels of JFM at different stages. While it is easy to recognize the incorrect reconstructed images in the last experiment, it is very difficult to notice any abnormality in the form of radial kernels and the 2D view of kernel functions in the first two experiments since the plots that correspond to JFM in Figures 1 and 2 seem correct. This fact demonstrates the importance of correct mathematical development and equations in the theory of image moments. Note that when J_n^B is used instead of J_n^A, the interleaving difference in sign will disappear due to equation (31) but the missing multiplicative term still remains.

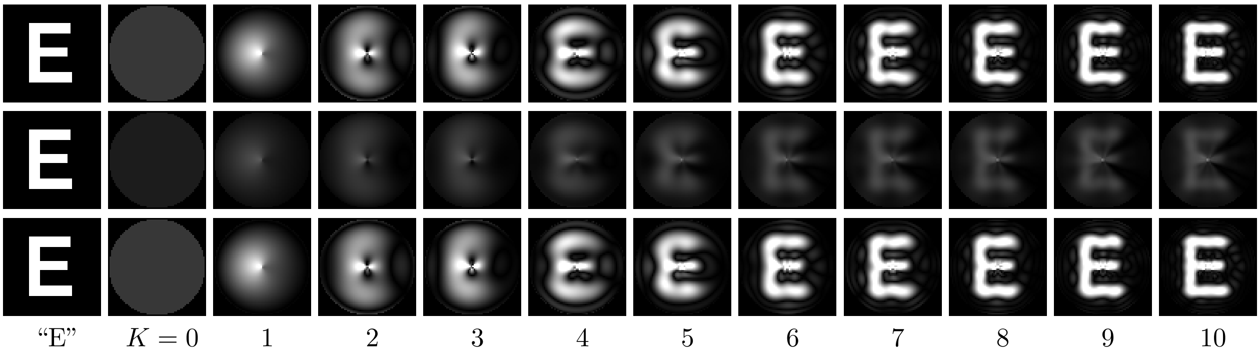


Figure 3: The reconstructed images for the character image "E" of size 64×64 for $K = 0, 1, \dots, 10$ by JFM, JFM$^{\mathrm{A}}$ for the case $p = 2, q = 2$ and the corresponding method OFMM. Top row: OFMM according to [6]. Middle row: JFM according to [11]. Bottom row: JFM$^{\mathrm{A}}$ redeveloped in this paper.

7 Conclusions

This paper has pointed out some confusing points in the definition of the radial kernels of Jacobi–Fourier moments in [11]. This confusion comes from the two related definitions of shifted Jacobi polynomials in the literature and results in some wrong analytical results in [11]. By using step-by-step development of relevant formulas, incorrect equations and expressions in [11] have been corrected in this paper, along with some other comments relating to the subject. Some experimental results on the radial functions, kernel functions, and reconstructed images demonstrate clearly the correctness of the formula developed in this paper. We believe that, in order to avoid this type of confusion, [11] should use only one definition of shifted Jacobi polynomials. Moreover, if G_n^B is used, it should be defined based on Gamma functions as given in this paper in order to make Jacobi–Fourier moments defined at non-integer values of p and q.

References

[1] M. R. Teague, "Image analysis via the general theory of moments," *J. Opt. Soc. Am.*, vol. 70, no. 8, pp. 920–930, 1980.

[2] A. B. Bhatia and E. Wolf, "On the circle polynomials of Zernike and related orthogonal sets," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 50, pp. 40–48, 1954.

[3] C.-H. Teh and R. Chin, "On image analysis by the methods of moments," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 10, no. 4, pp. 496–513, 1988.

[4] Y. Abu-Mostafa and D. Psaltis, "Recognitive aspects of moment invariants," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 6, no. 6, pp. 698–706, 1984.

[5] A. Khotanzad and Y. H. Hong, "Invariant image recognition by Zernike moments," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 12, no. 5, pp. 489–497, 1990.

[6] Y. Sheng and L. Shen, "Orthogonal Fourier–Mellin moments for invariant pattern recognition," *J. Opt. Soc. Am. A*, vol. 11, no. 6, pp. 1748–1757, 1994.

[7] J. Flusser, T. Suk, and B. Zitová, *Moments and Moment Invariants in Pattern Recognition.* John Wiley & Sons, 2009.

[8] T. V. Hoang and S. Tabbone, "Generic polar harmonic transforms for invariant image description," in *Proceedings of the 18th IEEE International Conference on Image Processing*, pp. 845–848, 2011.

[9] T. V. Hoang and S. Tabbone, "Fast computation of orthogonal polar harmonic transforms," in *Proceedings of the 21th International Conference on Pattern Recognition*, pp. 3160–3163, 2012.

[10] Q. Wang, O. Ronneberger, and H. Burkhardt, "Rotational invariance based on Fourier analysis in polar and spherical coordinates," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 31, no. 9, pp. 1715–1722, 2009.

[11] Z. Ping, H. Ren, J. Zou, Y. Sheng, and W. Bo, "Generic orthogonal moments: Jacobi–Fourier moments for invariant image description," *Pattern Recognition*, vol. 40, no. 4, pp. 1245–1254, 2007.

[12] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover Publications, 1964.

[13] F. Zernike, "Beugungstheorie des schneidenver-fahrens und seiner verbesserten form, der phasenkontrastmethode," *Physica*, vol. 1, no. 7-12, pp. 689 – 704, 1934.

[14] Z. Ping, R. Wu, and Y. Sheng, "Image description with Chebyshev–Fourier moments," *J. Opt. Soc. Am. A*, vol. 19, no. 9, pp. 1748–1754, 2002.

[15] G. Amu, S. Hasi, X. Yang, and Z. Ping, "Image analysis by pseudo-Jacobi ($p = 4, q = 3$)–Fourier moments," *Appl. Opt.*, vol. 43, no. 10, pp. 2093–2101, 2004.

[16] T. V. Hoang, *Image Representations for Pattern Recognition.* PhD thesis, University of Nancy, 2011.